MTH 295
Fall 2019
Homework 6
Due Thursday, 10/24 Name: Key

1) Let's find the general solution of the equation $y''-3y'+2y=3e^{-x}-10\cos 3x$.

a) First find the general solution of the homogeneous equation.

y_1, $y=e^{kx}$

$k^2-3k+2=0$

$(k-2)(k-1)=0$

$k=1,2$

$\boxed{y_g=c_1e^{x}+c_2e^{2x}}$

b) Write down the general form of the particular solution y_p of the non-homogeneous equation as well as its first two derivatives y_p' and y_p'' (you should have three unknowns, call them A, B, and C).

$y_p=Ae^{-x}+B\cos 3x+C\sin 3x$

$y_p'=-Ae^{-x}-3B\sin 3x+3C\cos 3x$

$y_p''=Ae^{-x}-9B\cos 3x-9C\sin 3x$

c) Find the values of A, B, and C and determine the particular solution of the non-homogeneous equation.

If y_p is a solution then

$Ae^{-x} - 9B\cos 3x - 9C\sin 3x + 3Ae^{-x} + 9B\sin 3x - 9C\cos 3x + 2Ae^{-x} + 2B\cos 3x + 2C\sin 3x = 3e^{-x} - 10\cos 3x$

so $A + 3A + 2A = 3$

$A = \frac{1}{2}$

$-9B - 9C + 2B = -10$

$-9C + 9B + 2C = 0$

or $7B + 9C = 10$

$9B - 7C = 0$

so $C = \frac{9}{7}B$

$7B + \frac{81}{7}B = 10$

$\frac{130}{7}B = 10$

$B = \frac{7}{13}$

and $C = \frac{9}{7} \cdot \frac{7}{13}$

$C = \frac{9}{13}$

so $\boxed{y_p = \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x}$

d) Using the results above, write the general solution of the non-homogeneous equation.

$y = y_g + y_p$

$\boxed{y = C_1 e^{x} + C_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x}$

2) For each of the following equations, give the form of y_p, a particular solution of the non-homogeneous equation, if the method of undetermined coefficients is to be used. Please use A, B, C, ... to denote your undetermined coefficients. Do not compute y_p, just give its form.

a) $y'' - 5y' + 6y = 6e^{5x}$

$k^2 - 5k + 6 = 0$

$(k-2)(k-3) = 0$

$k = 2, 3$

so $6e^{5x}$ is independent and

$\boxed{y_p = Ae^{5x}}$

b) $y'' - 7y' + 12y = x^5 e^{4x}$

$k^2 - 7k + 12 = 0$

$(k-3)(k-4) = 0$

so e^{4x} is dependent and

$\boxed{y_p = x(Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F)e^{4x}}$

c) $y'' - y = e^{-x} - e^x + e^x \cos x$

$k^2 - 1 = 0$

$k = 1, -1$

so e^{-x} and e^x are both dependent and

$\boxed{y_p = Axe^{-x} + Bxe^{x} + e^{x}(C\cos x + D\sin x)}$

d) $y'' + 4y' + 8y = x^2 e^{-2x} \sin 2x + x e^{-2x} \cos 2x$

$k^2 + 4k + 8 = 0$

$k = \frac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i$

so $y_g = e^{-2x} \{ c_1 \cos(2x) + c_2 \sin 2x \}$

and $\boxed{y_p = x(Ax^2 + Bx + C) e^{-2x} \sin 2x + x(Dx^2 + Ex + F) e^{-2x} \cos 2x}$

e) $y'' + 4y' + 13y = x^3 e^{-2x} \sin 3x$

$k = \frac{-4 \pm \sqrt{16 - 52}}{2}$

$= -2 \pm 3i$

$y_g = c_1 e^{-2x} \cos 3x + c_2 e^{-2x} \sin 2x$

so

$\boxed{y_p = x\left[(Ax^3 + Bx^2 + Cx + D) e^{-2x} \cos 3x + (Ex^3 + Fx^2 + Gx + H) e^{-2x} \sin 3x\right]}$

3) Use the method of variation of parameters to find the general solution of

$y''-2y'+y=\frac{e^x}{2x}$. Notice that undetermined coefficients will not work here.

if $y_g = e^{kx}$ then

$k^2-2k+1=$

$(k-1)^2=0$

$k=1$ is a repeated root so

$y_g = c_1 e^x + c_2 x e^x$

Now assume $y_p = y_1 v_1 + y_2 v_2$

$= e^x v_1 + x e^x v_2$

Then $v_1 = -\int \frac{y_2 f(x)}{W(y_1,y_2)}dx$ and $v_2 = \int \frac{y_1 f(x)}{W(y_1,y_2)}dx$ where $f(x) = e^x/2x$, $y_1 = e^x$, and $y_2 = xe^x$

$W(y_1,y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix} = e^{2x}$.

so $v_1 = -\int \frac{xe^x}{e^{2x}} \cdot \frac{e^x}{2x} dx = -\int \frac{1}{2} dx = -\frac{x}{2}$

and $v_2 = \int \frac{e^x}{e^{2x}} \cdot \frac{e^x}{2x} dx = \frac{1}{2}\int \frac{dx}{x} = \frac{1}{2}\ln|x|$

so $y_p = e^x(-\frac{x}{2}) + xe^x(\frac{1}{2}\ln|x|)$

and $y = y_g + y_p$

$= c_1e^x + c_2xe^x - \frac{1}{2}xe^x + \frac{1}{2}xe^x \ln|x|$

$\boxed{y = c_1e^x + c_2xe^x + \frac{1}{2}xe^x\ln|x|}$

because $c_2xe^x - \frac{1}{2}xe^x$

$= (c_2 - \frac{1}{2})xe^x$

$= c_2 x e^x$